

PIECEWISE CONSTANT LOCAL MARTINGALES WITH BOUNDED NUMBERS OF JUMPS

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ABSTRACT. A piecewise constant local martingale M with boundedly many jumps is a uniformly integrable martingale if and only if M_∞^- is integrable.

1. MAIN THEOREM

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ denote a filtered probability space with $\bigcup_{t \geq 0} \mathcal{F}_t \subset \mathcal{F}$. In Section 2, we shall prove the following theorem.

Theorem 1. *Assume for some $N \in \mathbb{N}_0$ and some stopping times $0 \leq \rho_1 \leq \dots \leq \rho_N$ we have a local martingale M of the form*

$$M = \sum_{m=1}^N J_m \mathbf{1}_{[\rho_m, \infty[}, \quad \text{that is,} \quad M_t = \sum_{m=1}^N J_m \mathbf{1}_{\{t \geq \rho_m\}}, \quad t \geq 0, \quad (1.1)$$

where J_m is \mathcal{F}_{ρ_m} -measurable for each $m = 1, \dots, N$. If

$$\mathbf{E} \left[\liminf_{t \uparrow \infty} M_t^- \right] < \infty \quad (1.2)$$

then M is a uniformly integrable martingale.

In (1.2), we could replace the limit inferior by a limit since M only has finitely many jumps and hence converges to a random variable M_∞ . Hence, (1.2) is equivalent to $\mathbf{E}[M_\infty^-] < \infty$.

Corollary 2. *Suppose the notation and assumptions of Theorem 1 hold, but with (1.2) replaced by*

$$\mathbf{E} [M_t^-] < \infty, \quad t \geq 0.$$

Then M is a martingale.

Proof. Fix a deterministic time $T \geq 0$ and consider the local martingale $\widetilde{M} = M^T$; that is, \widetilde{M} is the local martingale M stopped at time T . Then \widetilde{M} satisfies the conditions of Theorem 1, with J_m replaced by $J_m \mathbf{1}_{\{\rho_m \leq T\}}$ for each $m = 1, \dots, N$. Hence, \widetilde{M} is a uniformly integrable martingale. Since T was chosen arbitrarily the assertion follows. \square

Jacod and Shiryaev (1998) prove the following special case of Theorem 1.

Proposition 3. *Fix $N \in \mathbb{N}_0$ and assume we have a discrete-time filtration $\mathfrak{G} = (\mathcal{G}_m)_{m=0,1,\dots,N}$ and a \mathfrak{G} -local martingale $Y = (Y_m)_{m=0,1,\dots,N}$. If $\mathbf{E}[Y_N^-] < \infty$ then Y is a \mathfrak{G} -uniformly integrable martingale.*

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Note that Proposition 3 follows from Theorem 1. Indeed, define the continuous-time process M and the filtration $(\mathcal{F}_t)_{t \geq 0}$ by $M_t = Y_{[t] \wedge N}$ and $\mathcal{F}_t = \mathcal{G}_{[t] \wedge N}$, respectively, where $[t]$ denotes the largest integer smaller than or equal to t . Then M is a local martingale as in (1.1), with N replaced by $N + 1$. To see this, set $\rho_m = m - 1$ and $J_m = Y_{m-1} - Y_{m-2}$ with $Y_{-1} := 0$, for each $m = 1, \dots, N + 1$. Applying Theorem 1 then yields Proposition 3.

2. PROOFS OF THEOREM 1

In the following, we will provide two proofs of Theorem 1. The first one assumes Proposition 3 is already shown and reduces the more general situation of Theorem 1 to the discrete-time setup of Proposition 3. The second proof does not assume Proposition 3, but instead provides a direct argument based on an induction.

Proof I, relying on Proposition 3. Let us set $\rho = 0$ and $\rho_{N+1} = \infty$ and let $(\tau_n)_{n \in \mathbb{N}}$ denote a localization sequence of M such that M^{τ_n} is a uniformly integrable martingale for each $n \in \mathbb{N}$. For any stopping time τ we may define a sigma algebra

$$\mathcal{F}_{\tau-} = \sigma(\{A \cap \{t < \tau\}, A \in \mathcal{F}_t, t \geq 0\} \cup \mathcal{F}_0).$$

Note that $\{\tau = \infty\} = \bigcap_{n \in \mathbb{N}} \{n < \tau\} \in \mathcal{F}_{\tau-}$.

Let us now define a filtration $\mathfrak{G} = (\mathcal{G}_m)_{m=0, \dots, N}$ and a process $Y = (Y_m)_{m=0, 1, \dots, N}$ by $\mathcal{G}_m = \mathcal{F}_{\rho_m} \vee \mathcal{F}_{\rho_{m+1}-}$ and $Y_m = M_{\rho_m}$, respectively. Note that Y is adapted to \mathfrak{G} . Next, let us define a sequence $(\sigma_n)_{n \in \mathbb{N}}$ of random times, each taking values in $\{0, \dots, N - 1, \infty\}$ by

$$\sigma_n = \sum_{m=0}^{N-1} m \mathbf{1}_{\{\rho_m \leq \tau_n < \rho_{m+1} < \infty\}} + \infty \mathbf{1}_{\bigcup_{m=0}^N \{\rho_m \leq \tau_n\} \cap \{\rho_{m+1} = \infty\}}.$$

Then, σ_n is a \mathfrak{G} -stopping time for each $n \in \mathbb{N}$ since

$$\{\sigma_n = m\} = \{\rho_m \leq \tau_n < \rho_{m+1} < \infty\} \in \mathcal{F}_{\rho_m} \vee \mathcal{F}_{\rho_{m+1}-} = \mathcal{G}_m, \quad m = 0, \dots, N - 1,$$

and, furthermore, $\lim_{n \uparrow \infty} \sigma_n = \infty$.

We now fix $n \in \mathbb{N}$ and prove that Y^{σ_n} is a \mathfrak{G} -martingale, which then yields that Y is a \mathfrak{G} -local martingale. To this end, we have, for each $m = 0, \dots, N$,

$$\begin{aligned} Y_m^{\sigma_n} &= \sum_{k=0}^{N-1} M_{\rho_m \wedge k} \mathbf{1}_{\{\sigma_n = k\}} + M_{\rho_m} \mathbf{1}_{\{\sigma_n = \infty\}} \\ &= \sum_{k=0}^{N-1} M_{\rho_m \wedge k} \mathbf{1}_{\{\rho_k \leq \tau_n < \rho_{k+1} < \infty\}} + M_{\rho_m} \mathbf{1}_{\bigcup_{k=0}^N \{\rho_k \leq \tau_n\} \cap \{\rho_{k+1} = \infty\}} \\ &= M_{\rho_m}^{\tau_n}, \end{aligned}$$

yielding $\mathbb{E}[|Y_m^{\sigma_n}|] < \infty$. Now, fix $m = 1, \dots, N$. First, for any $A \in \mathcal{F}_{\rho_{m-1}}$, we have

$$\mathbb{E}[Y_m^{\sigma_n} \mathbf{1}_A] = \mathbb{E}[M_{\rho_m}^{\tau_n} \mathbf{1}_A] = \mathbb{E}[M_{\rho_{m-1}}^{\tau_n} \mathbf{1}_A] = \mathbb{E}[Y_{m-1}^{\sigma_n} \mathbf{1}_A];$$

next, for any $t \geq 0$ and $A \in \mathcal{F}_t$, we have

$$\begin{aligned} \mathbb{E}[Y_m^{\sigma_n} \mathbf{1}_{A \cap \{t < \rho_m\}}] &= \mathbb{E}[M_{\rho_m}^{\tau_n} \mathbf{1}_{A \cap \{t < \rho_m\}}] = \mathbb{E}[M_t^{\tau_n} \mathbf{1}_{A \cap \{t < \rho_m\}}] = \mathbb{E}[M_{\rho_{m-1}}^{\tau_n} \mathbf{1}_{A \cap \{t < \rho_m\}}] \\ &= \mathbb{E}[Y_{m-1}^{\sigma_n} \mathbf{1}_{A \cap \{t < \rho_m\}}], \end{aligned}$$

yielding that $\mathbb{E}[Y_m^{\sigma_n} \mathbf{1}_A] = \mathbb{E}[Y_{m-1}^{\sigma_n} \mathbf{1}_A]$ for all $A \in \mathcal{G}_{m-1}$. Hence, Y is indeed a \mathfrak{G} -local martingale.

The assumptions of the theorem yield that $\mathbb{E}[Y_N^-] < \infty$; hence Y a \mathfrak{G} -uniformly integrable martingale by Proposition 3. Now, fix $t \geq 0$ and $A \in \mathcal{F}_t$. Then we get $\mathbb{E}[|M_t|] + \mathbb{E}[|M_\infty|] \leq 2 \sum_{m=0}^N \mathbb{E}[|N_m|] < \infty$ and

$$\mathbb{E}[M_\infty \mathbf{1}_A] = \sum_{m=0}^N \mathbb{E}[Y_N \mathbf{1}_{A \cap \{\rho_m \leq t < \rho_{m+1}\}}] = \sum_{m=0}^N \mathbb{E}[Y_m \mathbf{1}_{A \cap \{\rho_m \leq t < \rho_{m+1}\}}]$$

$$= \sum_{m=0}^N \mathbb{E}[M_t \mathbf{1}_{A \cap \{\rho_m \leq t < \rho_{m+1}\}}] = \mathbb{E}[M_t \mathbf{1}_A]$$

since $A \cap \{\rho_m \leq t < \rho_{m+1}\} \in \mathcal{G}_m$ for each $m = 0, \dots, N$. Hence, M is indeed a uniformly integrable martingale. \square

Proof II, relying on an induction argument. We proceed by induction over N . The case $N = 0$ is clear. Hence, let us assume the assertion is proven for some $N \in \mathbb{N}_0$ and consider the assertion with N replaced by $N + 1$. Let $(\tau_n)_{n \in \mathbb{N}}$ denote a corresponding localization sequence such that M^{τ_n} is a uniformly integrable martingale for each $n \in \mathbb{N}$.

Step 1: In the first step, we want to argue that the nondecreasing sequence $(\hat{\tau}_n)_{n \in \mathbb{N}}$, given by

$$\hat{\tau}_n = \tau_n \mathbf{1}_{\{\tau_n < \rho_1\}} + \infty \mathbf{1}_{\{\tau_n \geq \rho_1\}} \geq \tau_n,$$

is also a localization sequence for M . To this end, fix $k \in \mathbb{N}$ and consider the process

$$\widetilde{M} = (M - M^{\tau_k}) \mathbf{1}_{\{\tau_k \geq \rho_1\}}.$$

Then we have

$$\widetilde{M}^- \leq M^- + |M^{\tau_k}|;$$

hence

$$\mathbb{E} \left[\liminf_{t \uparrow \infty} \widetilde{M}_t^- \right] \leq \mathbb{E} \left[\liminf_{t \uparrow \infty} M_t^- \right] + \mathbb{E} [|M_\infty^{\tau_k}|] < \infty. \quad (2.1)$$

Next, we argue that \widetilde{M} is also a local martingale, again with localization sequence $(\tau_n)_{n \in \mathbb{N}}$. Indeed, for $n \in \mathbb{N}$, $t, h \geq 0$, and $A \in \mathcal{F}_t$ note that

$$\begin{aligned} \mathbb{E} [\widetilde{M}_{t+h}^{\tau_n} \mathbf{1}_A] &= \mathbb{E} [(M_{t+h}^{\tau_n} - M_{t+h}^{\tau_n \wedge \tau_k}) \mathbf{1}_{A \cap \{\rho_1 \leq \tau_k \leq t\}}] + \mathbb{E} [(M_{t+h}^{\tau_n} - M_{t+h}^{\tau_n \wedge \tau_k}) \mathbf{1}_{A \cap \{\rho_1 \leq \tau_k\} \cap \{\tau_k > t\}}] \\ &= \mathbb{E} [(M_t^{\tau_n} - M_t^{\tau_n \wedge \tau_k}) \mathbf{1}_{A \cap \{\rho_1 \leq \tau_k \leq t\}}] + \mathbb{E} [(M_{t+h}^{\tau_n \wedge \tau_k} - M_{t+h}^{\tau_n \wedge \tau_k}) \mathbf{1}_{A \cap \{\rho_1 \leq \tau_k\} \cap \{\tau_k > t\}}] \\ &= \mathbb{E} [\widetilde{M}_t^{\tau_n} \mathbf{1}_A], \end{aligned}$$

where we used the definition of \widetilde{M} , $\{\rho_1 \leq \tau_k \leq t\} \in \mathcal{F}_t$, $A \cap \{\rho_1 \leq \tau_k\} \cap \{\tau_k > t\} \in \mathcal{F}_{\tau_k}$, and the martingale property of M^{τ_n} . Alternatively, we could have observed that $\widetilde{M} = \int_0^\cdot \mathbf{1}_{\{\rho_1 \leq \tau_k < s\}} dM_s$ (using the fact that $\mathbf{1}_{\{\rho_1 \leq \tau_k\}} \mathbf{1}_{\tau_k, \infty[\cdot]}$ is bounded and predictable since it is adapted and left-continuous). Hence, \widetilde{M} is a local martingale of the form

$$\widetilde{M} = \sum_{m=2}^{N+1} (J_m \mathbf{1}_{\{\rho_1 \leq \tau_k < \rho_m\}}) \mathbf{1}_{\llbracket \rho_m, \infty \rrbracket},$$

satisfying (2.1), and the induction hypothesis yields that \widetilde{M} is a uniformly integrable martingale. This again yields that

$$M^{\hat{\tau}_k} = M^{\tau_k} + \widetilde{M}$$

is also a uniformly integrable martingale, proving the claim that $(\hat{\tau}_n)_{n \in \mathbb{N}}$ is a localization sequence for M .

Step 2: We want to argue that $M_t \in \mathcal{L}^1$ for each $t \in [0, \infty]$. To this end, fix $t \in [0, \infty]$ and note

$$\mathbb{E} [|M_t|] \leq \liminf_{n \uparrow \infty} \mathbb{E} [|M_t^{\hat{\tau}_n}|] \quad (2.2)$$

$$= \mathbb{E} [M_0] + 2 \liminf_{n \uparrow \infty} \mathbb{E} \left[\left(M_t^{\hat{\tau}_n} \right)^- \right] \quad (2.3)$$

$$\leq \mathbb{E} [M_0] + 2 \liminf_{n \uparrow \infty} \mathbb{E} \left[\left(M_\infty^{\hat{\tau}_n} \right)^- \right] \quad (2.4)$$

$$\leq \mathbb{E}[M_0] + 2\mathbb{E}[M_\infty^-] \quad (2.5)$$

$$< \infty. \quad (2.6)$$

Here, the inequality in (2.2) is an application of Fatou's lemma. The equality in (2.3) relies on the fact that for any uniformly integrable martingale X we have $\mathbb{E}[|X_t|] = \mathbb{E}[X_t^+] + \mathbb{E}[X_t^-] = \mathbb{E}[X_0] + 2\mathbb{E}[X_t^-]$. The inequality in (2.4) uses that $(M^{\hat{\tau}_n})^-$ is a uniformly integrable submartingale, thanks to Jensen's inequality, for each $n \in \mathbb{N}$. The inequality in (2.5) (which is, actually, an equality) uses the fact that $M_{\hat{\tau}_n} \in \{0, M_\infty\}$, for each $n \in \mathbb{N}$, by construction of the localization sequence $(\hat{\tau}_n)_{n \in \mathbb{N}}$. Finally, the inequality in (2.6) holds by assumption.

Step 3: We now argue that M is a uniformly integrable martingale. To this end, fix $t \geq 0$ and $A \in \mathcal{F}_t$. Observe that

$$\begin{aligned} \mathbb{E}[M_\infty \mathbf{1}_A] &= \lim_{n \uparrow \infty} \left(\mathbb{E}[M_\infty \mathbf{1}_{A \cap \{\hat{\tau}_n < \rho_1 < \infty\}}] + \mathbb{E}[M_\infty \mathbf{1}_{A \cap \{\hat{\tau}_n < \rho_1\} \cap \{\rho_1 = \infty\}}] + \mathbb{E}[M_\infty \mathbf{1}_{A \cap \{\hat{\tau}_n \geq \rho_1\}}] \right) \\ &= \lim_{n \uparrow \infty} \mathbb{E}[M_\infty^{\hat{\tau}_n} \mathbf{1}_{A \cap \{\hat{\tau}_n = \infty\}}] \end{aligned} \quad (2.7)$$

$$\begin{aligned} &= \lim_{n \uparrow \infty} \left(\mathbb{E}[M_\infty^{\hat{\tau}_n} \mathbf{1}_{A \cap \{\hat{\tau}_n > t\}}] - \mathbb{E}[M_\infty^{\hat{\tau}_n} \mathbf{1}_{A \cap \{t < \hat{\tau}_n < \infty\}}] \right) \\ &= \lim_{n \uparrow \infty} \mathbb{E}[M_t^{\hat{\tau}_n} \mathbf{1}_{A \cap \{\hat{\tau}_n > t\}}] \end{aligned} \quad (2.8)$$

$$= \mathbb{E}[M_t \mathbf{1}_A]. \quad (2.9)$$

We obtained the equality in (2.7) since $\hat{\tau}_n = \infty$ on the event $\{\hat{\tau}_n \geq \rho_1\}$, and since the first term on the left-hand side is zero by the dominated convergence theorem and the second one thanks to the form of M . In (2.8), we used the martingale property of $M^{\hat{\tau}_n}$ in the first term and the fact that $M_{\hat{\tau}_n} = 0$ on the event $\{\hat{\tau}_n < \infty\}$ in the second term, for each $n \in \mathbb{N}$. Finally, we exchanged limit and expectation in (2.9) again by an application of the dominated convergence theorem. This then concludes the proof. \square

3. TWO EXAMPLES CONCERNING THE ASSUMPTIONS IN THEOREM 1

Example 4. Assume $(\Omega, \mathcal{F}, \mathbb{P})$ allows for a sequence $(\theta_m)_{m \in \mathbb{N}}$ of independent random variables with $\mathbb{P}[\theta_1 = 2] = 1$ and $\mathbb{P}[\theta_m = -1] = 1/2 = \mathbb{P}[\theta_m = 1]$ for all $m \geq 2$. Fix families $(J_m)_{m \in \mathbb{N}}$ and $(\rho_m)_{m \in \mathbb{N}}$ of random variables with

$$J_m = 2^{m-2} \theta_m \quad \text{and} \quad \rho_m = (1 - 1/m) \mathbf{1}_{\bigcap_{k=2}^{m-1} \{\theta_k = 1\}} + \infty \mathbf{1}_{\bigcup_{k=2}^{m-1} \{\theta_k = -1\}}.$$

Next, define M as in (1.1) with $N = \infty$ and assume that $(\mathcal{F}_t)_{t \geq 0}$ is the filtration generated by M . Then M is a local martingale, with localization sequence $(\rho_m)_{m \in \mathbb{N}}$. Indeed, M is a process that starts in one, and then, at times $1/2, 2/3, \dots$ doubles its value or jumps to zero, each with probability $1/2$. Since it eventually jumps to zero as $\mathbb{P}[\bigcup_{m=2}^{\infty} \{\theta_m = -1\}] = 1$, we have $M_1 = 0$. In particular, M is not a true martingale, but satisfies $\mathbb{E}[M_1^-] = 0 < \infty$. Thus, the assertions of Theorem 1 or Corollary 2 are not valid if $N = \infty$, even if $\mathbb{P}[\bigcup_{m \in \mathbb{N}} \{\rho_m = \infty\}] = 1$. \square

The next example illustrates that the assumptions of Corollary 2 are not sufficient to guarantee that M is a uniformly integrable martingale, even if there is only one jump possible, that is, even if $N = 1$. The example is adapted from Ruf (2015), where it is used as a counterexample for a different conjecture.

Example 5. Let ρ be an $\mathbb{N} \cup \{\infty\}$ -valued random variable with

$$\mathbb{P}[\rho = i] = \frac{1}{2i^2}, \quad i \in \mathbb{N}.$$

This then yields that

$$\mathbb{P}[\rho = \infty] = 1 - \frac{\pi^2}{12}.$$

Moreover, let θ be an independent $\{-1, 1\}$ valued random variable with $\mathbb{P}[\theta = 1] = \mathbb{P}[\theta = -1] = 1/2$. Define $J = \theta\rho^2$. Then the stochastic process

$$M = J\mathbf{1}_{[\rho, \infty[},$$

along with the filtration $(\mathcal{F}_t)_{t \geq 0}$ it generates, satisfies exactly the conditions of Corollary 2. Indeed, ρ is an \mathfrak{F} -stopping time and $M_t^- \leq \rho^2 \mathbf{1}_{\{\rho \leq t\}} \leq t^2$, hence $M_t^- \in \mathcal{L}^1$ for each $t \geq 0$. Thus, M is a martingale. This fact would also be very easy to check by hand.

We have $M_\infty = \lim_{t \uparrow \infty} M_t$ exists and satisfies $|M_\infty| = \rho^2 \mathbf{1}_{\{\rho < \infty\}}$. Thus,

$$\mathbb{E}[|M_\infty|] = \sum_{i \in \mathbb{N}} i^2 \frac{1}{2i^2} = \infty,$$

and M cannot be a uniformly integrable martingale. □

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